

# Deriving Practical Implementations of First-Class Functions

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# Intermediate theories

## **Reduction Theory**

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What do we know about our implementations? Are they correct?

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Reductions are often **compatible**, thereby rewrites can be applied in any order.

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A variable is **free** if there is no binder for it.

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**Substitution**,  $M[N/x]$ , replaces  $N$  for a variable  $x$  in  $M$ .

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Can reduce fewer terms than call-by-name, e.g.

$$(\lambda x. 42) \Omega$$



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Call-by-need will share the computation of  $(1 + 3)$ .

## Call-by-need $\lambda$ -calculus

Applied  $\lambda$ -expressions are like let-expressions:

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Instead of  $\beta$ , the call-by-need calculus operates on graphs represented by let-expressions.

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- Explicit substitutions completely remove substitutions from the meta language
- Abstract machines avoid the evaluation context's meta operation

# Operational Semantics

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An operational semantics describes a method for computing a result.

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Evaluation chains multiple steps till a result is found:

$$\text{eval}(M) = M' \quad \text{where } M \mapsto^* M' \not\mapsto M''$$

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The only difference in these two strategies is forcing the argument.

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$Eval\ Context_{\mathcal{L}} ::= \square \mid E N \mid \text{let } x := N \text{ in } E \mid \text{let } x := E \text{ in } E[x]$

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This style of semantics can be easily implemented as a recursive traversal of a term.



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$$\frac{\langle \Phi \parallel M \rangle \Downarrow_{\mathcal{L}} (\Phi', \lambda x. L) \quad \langle \Phi', x' \mapsto N \parallel L[x'/x] \rangle \Downarrow_{\mathcal{L}} R}{\langle \Phi \parallel M N \rangle \Downarrow_{\mathcal{L}} R}$$

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$$\frac{\Sigma(x) = (\Sigma', M) \quad \langle \Sigma' \parallel M \rangle \Downarrow_{\mathcal{N}} R}{\langle \Sigma \parallel x \rangle \Downarrow_{\mathcal{N}} R}$$

Evaluation of variable is delayed until lookup.

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Call-by-value does **not** require thunk closures.

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# Combinators and their Machines

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Combinator machines (Turner) take combinators to be the only function primitives.

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Notice the lack of  $\beta$ -reduction and variables.



# Super-combinators and Lambda-lifting

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e.g.

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generates the super-combinator:

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And a partial applications is left in place of the original function:

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Combinator machines approach has been largely abandoned for environment machines.

# Abstract Machines

## What are abstract machines?

Recall the evaluation rule for small-step semantics:

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Instead of searching for the next redex, abstract machines keep track of the context as state.

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# Curried Functions in Machines

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The closure  $((E, x \mapsto 1), \lambda y. x + y)$  is unnecessary!

# Multi-arity Function Calls

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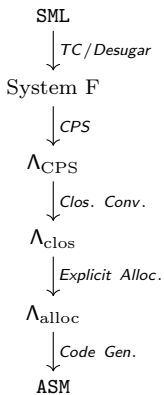
The differences in evaluation strategy:

- The number of closures
- Update stacks for call-by-need

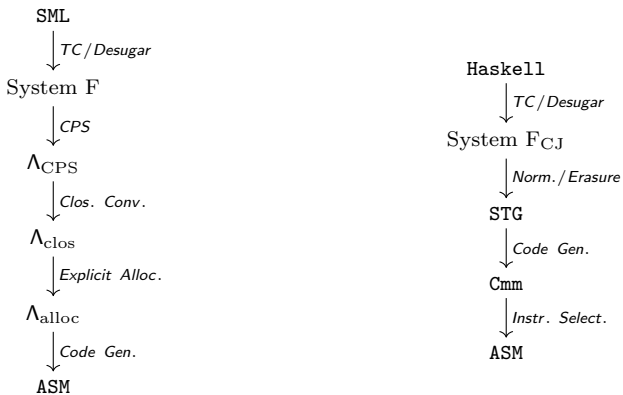
# Compilation through Intermediate Languages

The structure of modern functional compilers involves several intermediate languages:

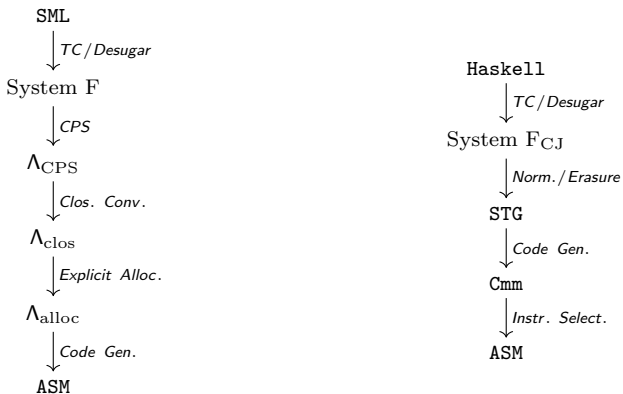
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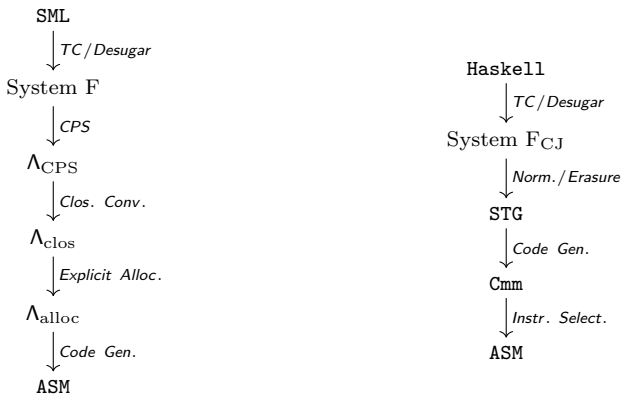


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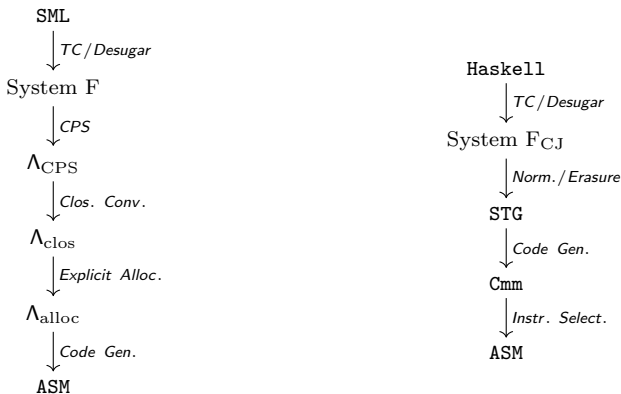


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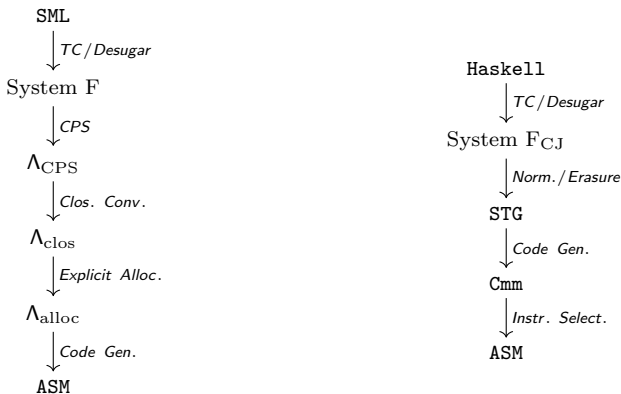
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- **Generality**, can be used to unify evaluation strategies (thinking, call-by-push-value)

# Closure-conversion

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**Closure-conversion** turns the former into the latter.

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- Call-by-need implementations require “heaps” and update frames.

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# Thanks



# Reasoning about Implementations

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- Reduction theory

- Operational Semantics

- Combinator and abstract machines

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How do we know that these are equivalent methods?



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Only shows that machine states respect the source; nothing about whether equalities of the source are preserved.

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Type preservation can give us a typing derivation in our target language

- Help us prove things like strong normalization
- Type information can inform code-generation and runtime systems

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We found that call-by-name reasoning easily adapted to call-by-value logical relations.

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There are still open questions regarding memoizing heaps:

- Objects are updated within according to the semantics
- In many C dynamic memory, these are unordered structures
- Applies to both delay-force and call-by-need